CONCERNING CELLULAR DECOMPOSITIONS OF 3-MANIFOLDS WITH BOUNDARY

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1. **Introduction.** In [2], we proved that if G is a cellular decomposition of a 3-manifold M such that the associated decomposition space is a 3-manifold N, then M and N are homeomorphic. In this paper we shall establish a related result for 3-manifolds with boundary. We shall show that if G is a cellular decomposition of a 3-manifold with boundary M such that the associated decomposition space is a 3-manifold with boundary N, then M and N are homeomorphic.

The techniques of this paper have applications to the study of embeddings of curves and surfaces in 3-manifolds. Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that the associated decomposition space is a 3-manifold with boundary N. By Theorem 2 of this paper, M and N are homeomorphic. Suppose K is a surface or a curve in M such that no nondegenerate element of G intersects K. If P denotes the projection map from M onto N, then $P|K$ is a homeomorphism. It is natural to ask the following: Is $P[K]$ embedded in N the same way that K is embedded in M? In §5, we give an affirmative answer to this question. In particular, K is tame if and only if $P[K]$ is tame.

In §6 we shall show that if G is a cellular decomposition of a 3-manifold with boundary into a 3-manifold with boundary, then the projection map can be approximated arbitrarily closely by homeomorphisms.

In [2], we established a theorem of basic importance in the study of cellular decompositions of 3-manifolds that yield 3-manifolds as their decomposition spaces. The main result, Theorem 1, of this paper is a useful corollary of the results of [2]. The results mentioned in the preceding two paragraphs are applications of Theorem 1. In [4], we shall apply Theorem 1 to the study of shrinkability conditions which are satisfied by certain cellular decompositions of E^3 that yield E^3 as their decomposition space.

2. **Terminology and notation.** The statement that M is a *3-manifold with boundary* means that M is a separable metric space such that each point of M has a neighborhood in M which is a 3-cell. If M is a 3-manifold with boundary, a point p of M is an *interior point* of M if and only if p has an open neighborhood in M which is an open 3-cell. The *interior* of M, Int M, is the set of all boundary points,

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and the *boundary* of M, Bd M, is $M-\text{Int } M$. The statement that M is a *3-manifold* means that M is a 3-manifold with boundary such that Bd M is void.

If M is a 3-manifold with boundary, a set X in M is a *cellular* subset of M if and only if there exists a sequence $C_1, C_2, C_3, \ldots$ of 3-cells in M such that (1) for each positive integer n, $C_{n+1} \subset \text{Int } C_n$, and (2) $X = \bigcap_{i=1}^{\infty} C_i$. Note that a cellular subset of a 3-manifold with boundary M necessarily lies in the interior of M.

If M is a 3-manifold with boundary, the statement that G is a *cellular decomposition* of M means that G is an upper semicontinuous decomposition of M such that each element of G is a cellular subset of M.

If X is a topological space and G is an upper semicontinuous decomposition of X, then X/G denotes the associated decomposition space, P denotes the projection map from X onto X/G, and H_G denotes the union of all the nondegenerate elements of G.

If A is a set in a topological space X, let βA denote the (topological) boundary of A, and let Cl A denote the closure of A. If X is a metric space, $A \subset X$, and ε is a positive number, then $V(\varepsilon, A)$ denotes the open ε-neighborhood of A.

3. **The main result.** The purpose of this section is to establish the main result of the paper. We shall depend heavily on the following result from [2].

THEOREM 1 OF [2]. *Suppose that M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N. Suppose that N has a triangulation T such that if σ is a simplex of T, $P^{-1}[\sigma]$ lies in an open 3-cell U_σ in M. Then there exists a triangulation Σ of M and an isomorphism ϕ from T onto Σ such that for each simplex σ of T, $\phi(\sigma) \subset U_\sigma$.*

THEOREM 1. *Suppose M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Suppose U is an open set in* Int N *such that βU misses $P[H_G]$. Then there is a homeomorphism h from* Cl $P^{-1}[U]$ *onto* Cl U *such that $h|\beta P^{-1}[U] = P|\beta P^{-1}[U]$.*

Proof. We shall apply Theorem 1 of [2], and we have some preliminary steps to take before we can apply Theorem 1 of [2].

For each positive integer k, let A_k be the union of all the sets of G lying in $P^{-1}[U]$ and of diameter at least $1/k$. Then by upper semicontinuity of G, A_k is closed. It follows that there exists a sequence $V_1, V_2, V_3, \ldots$ of open sets in M such that for each positive integer i, V_i contains $\beta P^{-1}[U]$, $\overline{V}_{i+1} \subset V_i$, $\overline{V}_i$ and A_i are disjoint, and $V_i \subset V(1/i, \beta P^{-1}[U])$. There is an open covering $\mathscr{W}$ of $P^{-1}[U]$ such that (1) if $W \in \mathscr{W}$, W is an open 3-cell and $\overline{W} \subset P^{-1}[U]$, (2) if n is any positive integer, W is a set of $\mathscr{W}$, and W intersects $\overline{V}_{n+1}$, then (diam W) $< 1/n$, and (3) if $g \in G$ and $g \subset P^{-1}[U]$, then there is a set W of $\mathscr{W}$ such that $g \subset W$. Such a set $\mathscr{W}$ may be constructed in the following way. If $g \in G$ and $g \subset A_1$, there is an open 3-cell W_g such that $g \subset W_g$, $\overline{W}_g \subset P^{-1}[U]$, and W_g and $\overline{V}_1$ are disjoint. If $g \in G$ and $g \subset A_2 - A_1$, there is an open 3-cell W_g such that $g \subset W_g$, $\overline{W}_g \subset P^{-1}[U]$, (diam W_g) < 1, and W_g and $\overline{V}_2$ are disjoint. If n is any positive integer, $n > 1$, $g \in G$, and $g \subset A_n - A_{n-1}$, there is an open 3-cell W_g such that $g \subset W_g$, $\overline{W}_g \subset P^{-1}[U]$, (diam W_g) $< 1/(n-1)$, and W_g and $\overline{V}_n$ are disjoint. Let $\mathscr{W}$ be the collection of all such sets W_g for the elements g of G lying in $P^{-1}[U]$.

We shall show that if n is any positive integer, W is a set of $\mathscr{W}$, and W intersects $\overline{V}_{n+1}$, then (diam W) $< 1/n$. If W intersects $\overline{V}_{n+1}$ and g is any set of G lying in A_{n+1}, then W is distinct from W_g. It follows that (diam W_g) $< 1/n$. The remaining properties of $\mathscr{W}$ are evident.

There is a triangulation T of U such that (1) if n is a positive integer, $\sigma \in T$, and σ intersects $P[V_n]$, then (diam σ) $< 1/n$, and (2) if $\sigma \in T$, then $P^{-1}[\sigma]$ lies in some set of $\mathscr{W}$. For each simplex σ of T, let W_σ be some open 3-cell of $\mathscr{W}$ such that $P^{-1}[\sigma] \subset W_\sigma$.

Let G_0 be the set of all elements of G contained in $P^{-1}[U]$. Then G_0 is a cellular decomposition of the 3-manifold $P^{-1}[U]$. By Theorem 1 of [2], there exist a triangulation Σ of $P^{-1}[U]$ and an isomorphism ϕ from T onto Σ such that if $\sigma \in T$, $\phi(\sigma) \subset W_\sigma$.

Since ϕ^{-1} is an isomorphism from Σ onto T, by the proof of Lemma 8 of [2], there is a homeomorphism h_0 from $P^{-1}[U]$ onto U such that if σ is a simplex of Σ, then $h_0[\sigma] = \phi^{-1}(\sigma)$. Define a function h as follows: (1) If $x \in P^{-1}[U]$, then $h(x) = h_0(x)$. (2) If $x \in \beta P^{-1}[U]$, then $h(x) = P(x)$. Since $P^{-1}[\beta U] = \beta P^{-1}[U]$, and $\beta P^{-1}[U]$ and $P^{-1}[U]$ are disjoint, h is well defined. Clearly h is from Cl $P^{-1}[U]$, and since h_0 is onto U and $P|\beta P^{-1}[U]$ is onto βU, h is onto Cl U. By definition of h, $h|\beta P^{-1}[U] = P|\beta P^{-1}[U]$. In order to complete the proof of Theorem 1, we need only to show that h is a homeomorphism.

Since βU misses $P[H_G]$, P is one-to-one on $\beta P^{-1}[U]$. Since h_0 is one-to-one, it follows that h is one-to-one.

Now we shall prove that h is continuous. Clearly, it is sufficient to show that if $x_1, x_2, x_3, \ldots$ is a sequence of points in $P^{-1}[U]$ and converging to the point x_0 of $\beta P^{-1}[U]$, then $h(x_1), h(x_2), h(x_3), \ldots$ converges to $h(x_0)$, or, in view of the definition of h, to $P(x_0)$.

For each positive integer i, let τ_i be a 3-simplex of Σ containing x_i, and let σ_i be $\phi^{-1}(\tau_i)$. By construction of h, $h(x_i) \in \sigma_i$. Now $x_1, x_2, x_3, \ldots$ converges to x_0 and $x_0 \in \beta P^{-1}[U]$. Further, for each positive integer i, $\tau_i \subset W_{\sigma_i}$. It follows from these facts and properties of $\mathscr{W}$ that (diam τ_1), (diam τ_2), (diam τ_3), ... converges to 0.

Let Q be a neighborhood of $h(x_0)$. Since $h(x_0) = P(x_0)$ and $\{x_0\}$ is an element of G, $P^{-1}[Q]$ is a neighborhood of x_0. From facts mentioned in the preceding paragraph, it follows that there is a positive integer s such that if $n > s$, $\tau_n \subset P^{-1}[Q]$. It follows from facts about the construction of $\mathscr{W}$ that there is a positive integer t greater than s such that if $n > t$, $W_{\sigma_n} \subset P^{-1}[Q]$. Hence if $n > t$, $P[W_{\sigma_n}] \subset Q$. The open covering $\mathscr{W}$ has the property that if $\sigma \in T$, $P^{-1}[\sigma] \subset W_\sigma$. Hence for each positive integer n, $P^{-1}[\sigma_n] \subset W_{\sigma_n}$, and if $n > t$, then both $\sigma_n \subset P[W_{\sigma_n}] \subset Q$ and $h(x_n) \in Q$. It follows that $h(x_1), h(x_2), h(x_3), \ldots$ converges to $h(x_0)$. Consequently h is continuous.

Now we shall prove that h^{-1} is continuous. It is sufficient to show that if

$y_1, y_2, y_3, \ldots$ is a sequence of points in U converging to the point y_0 of βU, then $h^{-1}(y_1), h^{-1}(y_2), h^{-1}(y_3), \ldots$ converges to $h^{-1}(y_0)$, or equivalently, to $P^{-1}[y_0]$.

For each positive integer i, let σ_i be a 3-simplex of T containing y_i, and let τ_i be $\phi(\sigma_i)$. By construction of h, for each i, $h^{-1}(y_i) \in \tau_i$.

Suppose R is a neighborhood of $P^{-1}[y_0]$. Since $\{P^{-1}[y_0]\}$ is an element of G, there is a neighborhood R_0 of $P^{-1}[y_0]$ such that $R_0 \subset R$, R_0 is a union of elements of G, and if $W \in \mathscr{W}$ and W intersects R_0, then $W \subset R$. Notice that $P[R_0]$ is a neighborhood in N of y_0.

There is a positive integer t such that if $n > t$, $y_n \in P[R_0]$. If $n > t$, $P^{-1}[y_n] \in R_0$. Since for each i, $P^{-1}[\sigma_i] \subset W_{\sigma_i}$ and $y_i \in \sigma_i$, then if $i > t$, W_{σ_i} intersects R_0 and hence lies in R. By construction of Σ, it follows that for each i, $\tau_i \subset W_{\sigma_i}$, and hence if $i > t$, $\tau_i \subset R$ and therefore $h^{-1}(y_i) \in R$. It follows that $h^{-1}(y_1), h^{-1}(y_2), h^{-1}(y_3), \ldots$ converges to $P^{-1}[y_0]$, or to $h^{-1}(y_0)$. Hence h^{-1} is continuous.

Therefore h is a homeomorphism from Cl $P^{-1}[U]$ onto Cl U such that $h|\beta P^{-1}[U] = P|\beta P^{-1}[U]$. This establishes Theorem 1.

4. **Application to 3-manifolds with boundary.** We are now prepared to extend Theorem 2 of [2] to the case of cellular decompositions of 3-manifolds with boundary.

THEOREM 2. *Suppose M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Then there is a homeomorphism h from M onto N such that $h|$Bd $M = P|$Bd M.*

Proof. Int N is an open subset of N lying in Int N, and β(Int N) = Bd N. Further, P^{-1}[Int N] = Int M, and $P^{-1}[\beta$(Int N)] = Bd M. We also have that Cl Int $M = M$ and Cl Int $N = N$. With the aid of Theorem 1, it follows that there exists a homeomorphism h from M onto N such that $h|$Bd $M = P|$Bd M.

5. **Applications to embeddings.** In this section we establish some results concerning embeddings of surfaces and curves in manifolds. Our first result is a slightly more general theorem.

THEOREM 3. *Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Suppose that K is a closed nowhere dense subset of M such that K and H_G are disjoint. Then there is a homeomorphism h from M onto N such that $h|K = P|K$.*

Proof. This follows by applying Theorem 1 to the open subset (Int M) $- K$ of M. Since K is nowhere dense in M, Cl [(Int M) $- K$] $= M$, and it follows that there exists a homeomorphism h from M onto N such that $h|(K \cup$ Bd $M) = P|(K \cup$ Bd $M)$. In particular, $h|K = P|K$.

COROLLARY 1. *Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Suppose K is a* *manifold with boundary, of dimension* 1 *or* 2, *contained in* M, *and missing* H_G. *Then there is a homeomorphism* h *from* M *onto* N *such that* $h|K=P|K$.

The next two corollaries may be regarded as extensions of Theorem 1 of [1].

COROLLARY 2. *Suppose that* M *is a* 3-*manifold with boundary and* G *is a cellular decomposition of* M *such that* M/G *is a* 3-*manifold with boundary. Suppose* K *is a* 2-*manifold with boundary in* M *such that* K *misses* H_G. *Then* $P[K]$ *is tame in* N *if and only if* K *is tame in* M.

A compact connected set X in E^3 is *pointlike* in E^3 if and only if E^3-X is homeomorphic to $E^3-\{0\}$. It is well known (see [6]) that in E^3, "pointlike" and "cellular" are equivalent. By a *pointlike decomposition* of E^3 is meant an upper semicontinuous decomposition of E^3 into pointlike compact connected sets.

COROLLARY 3. *Suppose that* G *is a pointlike decomposition of* E^3 *such that* E^3/G *is homeomorphic to* E^3. *Suppose* K *is a* 2-*manifold in* E^3 *such that* K *misses* H_G. *Then* $P[K]$ *is tame if and only if* K *is tame.*

COROLLARY 4. *Suppose that* G *is a pointlike decomposition of* E^3 *such that* E^3/G *is homeomorphic to* E^3. *Suppose that* J *is an arc or a simple closed curve in* E^3 *such that* J *misses* H_G. *Then* $P[J]$ *is tame if and only if* J *is tame.*

It follows from results of [3] and [5] that under the hypothesis of Corollary 4, if J is a simple closed curve, $\pi_1(E^3-J)$ and $\pi_1(E^3-P[J])$ are isomorphic. Corollary 1 gives a considerably stronger result in this case.

6. **Approximating the projection map.** The result of this section shows that if G is a cellular decomposition of a 3-manifold with boundary into a 3-manifold with boundary, then the projection map can be approximated arbitrarily closely by homeomorphisms. D. R. McMillan, Jr. raised the question as to whether such approximations are possible.

THEOREM 4. *Suppose that* M *is a* 3-*manifold with boundary and* G *is a cellular decomposition of* M *such that* M/G *is a* 3-*manifold with boundary* N. *Suppose* U *is an open set in* Int M *containing* H_G *and* ε *is a positive number. Then there exists a homeomorphism* h *from* M *onto* N *such that* (1) *if* $x \in M-U$, $h(x)=P(x)$ *and* (2) *if* $x \in M$, $d(h(x), P(x))<\varepsilon$.

Proof. The proof of this theorem is a modification of the proof of Theorem 1 above. Let $\mathscr{A}$ be an open covering of N by sets of diameter less than $\varepsilon/2$. Let $V_1, V_2, V_3, \ldots$ be as in the proof of Theorem 1. There is an open covering $\mathscr{W}$ of U such that (1) if $W \in \mathscr{W}$, W is an open 3-cell and $\overline{W} \subset U$, (2) if n is any positive integer, $W \in \mathscr{W}$, and W intersects $\overline{V}_{n+1}$, then (diam $W)<1/n$, (3) if $g \in G$ and $g \subset P^{-1}[U]$, then g lies in some set of $\mathscr{W}$, and (4) if $W \in \mathscr{W}$, there is a set A of $\mathscr{A}$ such that $P[W] \subset A$.

Since $H_G \subset U$, $P[U]$ is open in N. There is a triangulation T of $P[U]$ such that (1) if n is any positive integer, $\sigma \in T$, and σ intersects $P[V_n]$, then (diam σ) $< 1/n$, and (2) if $\sigma \in T$, then $P^{-1}[\sigma]$ lies in some set of $\mathscr{W}$. For each simplex σ of T, let W_σ be some open 3-cell of $\mathscr{W}$ such that $P^{-1}[\sigma] \subset W_\sigma$.

Let G_0 be the set of all elements of G contained in U. Then G_0 is a cellular decomposition of the 3-manifold U. By Theorem 1 of [2], there exist a triangulation Σ of U and an isomorphism ϕ from T onto Σ such that if $\sigma \in T$, $\phi(\sigma) \subset W_\sigma$. By the proof of Lemma 8 of [2], there is a homeomorphism h_0 from U onto $P[U]$ such that if $\sigma \in \Sigma$, $h_0[\sigma] = \phi^{-1}(\sigma)$. Define a function h as follows: (1) If $x \in M - U$, $h(x) = P(x)$. (2) If $x \in U$, $h(x) = h_0(x)$. As in the proof of Theorem 1, we may show that h is a homeomorphism from M onto N. By definition, if $x \in M - U$, $h(x) = P(x)$.

We shall show now that if $x \in M$, $d(h(x), P(x)) < \varepsilon$. If $x \in M - U$, clearly $d(h(x), P(x)) < \varepsilon$. Suppose $x \in U$. Let σ be a 3-simplex of T containing $P(x)$, and let τ be a 3-simplex of T containing $h(x)$. We shall prove that $P[W_\sigma]$ and $P[W_\tau]$ intersect. First, since $P(x) \in \sigma$, $x \in P^{-1}[\sigma]$, and since $P^{-1}[\sigma] \subset W_\sigma$, then $x \in W_\sigma$. Second, since $h(x) \in \tau$, then by the way h is defined, $x \in \phi(\tau)$. Since $\phi(\tau) \subset W_\tau$, then $x \in W_\tau$. Hence $x \in W_\sigma \cap W_\tau$, and thus $P[W_\sigma]$ and $P[W_\tau]$ intersect.

By construction of $\mathscr{W}$, if $W \in \mathscr{W}$, then for some set A of $\mathscr{A}$, $P[W] \subset A$ and so (diam $P[W]) < \varepsilon/2$. Since $x \in W_\sigma$, then $P(x) \in P[W_\sigma]$. Since $P^{-1}[\tau] \subset W_\tau$, then $\tau \subset P[W_\tau]$; since $h(x) \in \tau$, $h(x) \in P[W_\tau]$. It follows that $d(h(x), P(x)) < \varepsilon$. This completes the proof of Theorem 4.

References

1. S. Armentrout, *Upper semicontinuous decompositions of E^3 with at most countably many nondegenerate elements*, Ann. of Math. **78** (1963), 605–618.

2. ———, *Cellular decompositions of* 3-*manifolds that yield* 3-*manifolds*, (to appear).

3. ———, *Homotopy properties of decomposition spaces*, (to appear).

4. ———, *Shrinkability of certain decompositions of E^3 that yield E^3*, Illinois J. Math. (to appear).

5. S. Armentrout and T. M. Price, *Decompositions into compact sets with UV properties*, (to appear).

6. D. G. Stewart, *Cellular subsets of the* 3-*sphere*, Trans. Amer. Math. Soc. **114** (1965), 10–22.

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